UNIT III

Isomorphic, Homomorphic and Sub Graphs

3.1 Isomorphic Graphs

Two graphs G and G' are said to be isomorphic if there is a one-to-one correspondence between their vertices and between their edges such that incidence relationship is preserved. In the other words, suppose that edge e is incident on vertices v1 and v2 in G; then the corresponding edge e' in G' must be incident on the vertices v1' and v2' that correspond to v1 and v2 respectively.

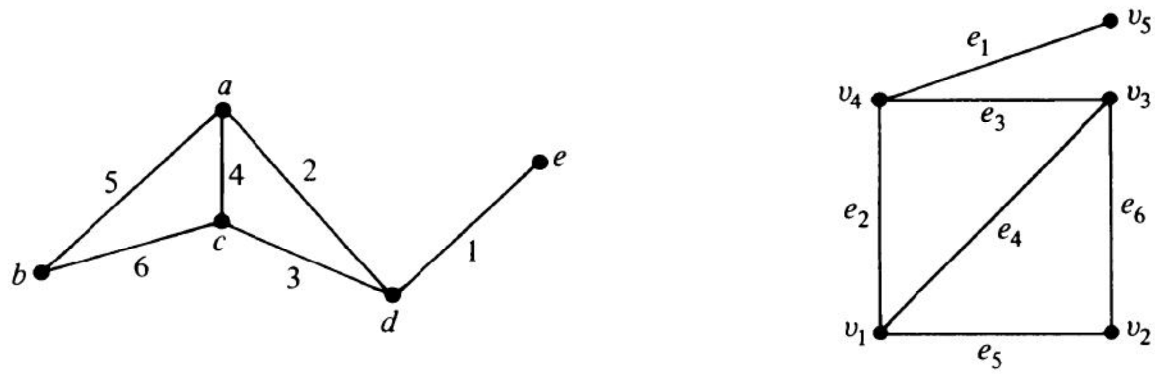


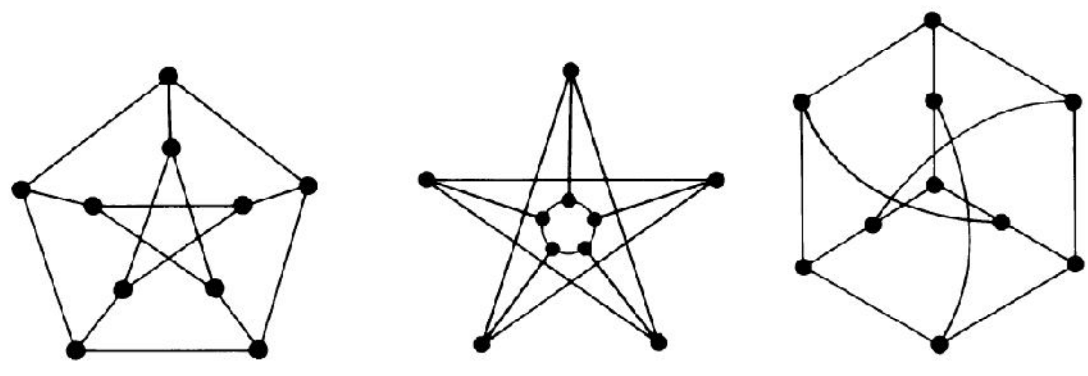
Fig 3.1

Two graphs in the figure above are isomorphic. The correspondence between the two graphs is as follow: the vertices a, b, c, d, and e correspond to v1, v2, v3, v4, and v5 respectively. The edges 1, 2, 3, 4, 5, and 6 correspond to e1, e2, e3, e4, e5, and e6 respectively.

Two isomorphic graphs must have

- The same number of vertices
- The same number of edges
- An equal number of vertices with a given degree

Example: Isomorphic Graph



(a) (b) (c)

Fig 3.2

Example: Non-isomorphic Graph

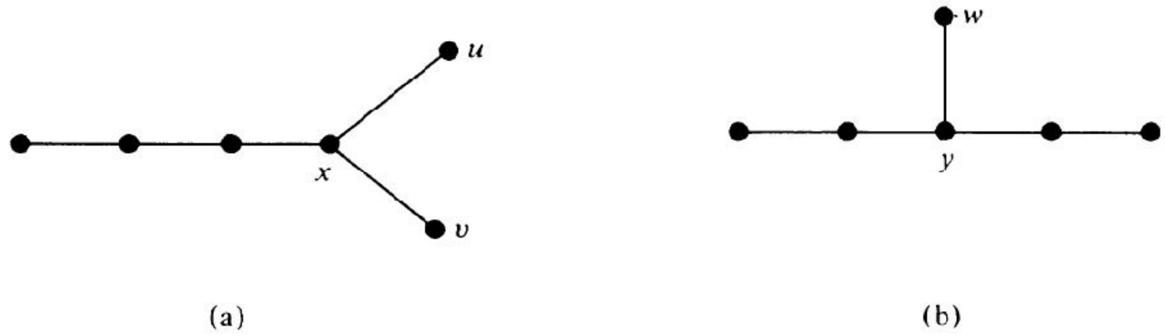


Fig 3.3

Two graphs showing above satisfy all three conditions but still they are not isomorphic. If the graph (a) is to be isomorphic to (b), vertex x must be correspond to y, because there are no other vertices of degree three. Now in (b) there is only one pendant vertex , w, adjacent to y, while in (a) there are two pendant vertices, u and v, adjacent to x.

A graph can exist in different forms having the same number of vertices, edges, and also the same edge connectivity. Such graphs are called isomorphic graphs.

Two graphs G1 and G2 are said to be isomorphic if:

- Their number of components (vertices and edges) are same.
- Their edge connectivity is retained.

An unlabelled graph also can be thought of as an isomorphic graph.

Theorem: [Isomorphism] Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $f : V_1 \rightarrow V_2$ that preserves the adjacency, i.e. $uv \in E_1$ if and only if $f(u)f(v) \in E_2$.

There exists a function ‘f’ from vertices of G1 to vertices of G2

[f: V(G1) V(G2)], such that

Case (i): *f* is a bijection (both one-one and onto)
Case (ii): *f* preserves adjacency of vertices, i.e., if the edge {U, V} ∈ G1, then the edge {f(U), f(V)} ∈ G2, then G1 ≡ G2.

If G1 ≡ G2 then:

1. |V(G1)| = |V(G2)|
2. |E(G1)| = |E(G2)|
3. Degree sequences of G1 and G2 are same.

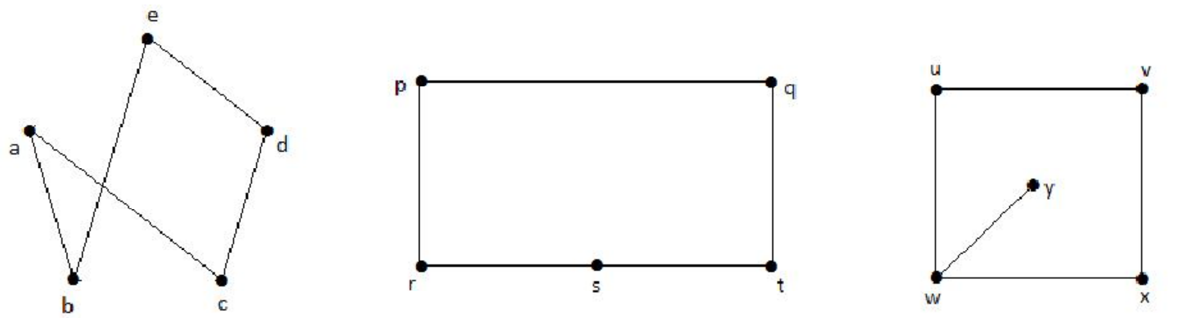
4. If the vertices {V1, V2, .. Vk} form a cycle of length K in G1, then the vertices {f(V1), f(V2),… f(Vk)} should form a cycle of length K in G2.

All the above conditions are necessary for the graphs G1 and G2 to be isomorphic, but not sufficient to prove that the graphs are isomorphic.

- (G1 ≡ G2) if and only if (G1’ ≡ G2’) where G1 and G2 are simple graphs.
- (G1 ≡ G2) if the adjacency matrices of G1 and G2 are same.
- (G1 ≡ G2) if and only if the corresponding subgraphs of G1 and G2 (obtained by deleting some vertices in G1 and their images in graph G2) are isomorphic.

Example

Which of the following graphs are isomorphic?



G1 G2 G3

Fig 3.4

In the graph G3, vertex ‘w’ has only degree 3, whereas all the other graph vertices has degree 2. Hence G3 not isomorphic to G1 or G2.

Taking complements of G1 and G2, you have:

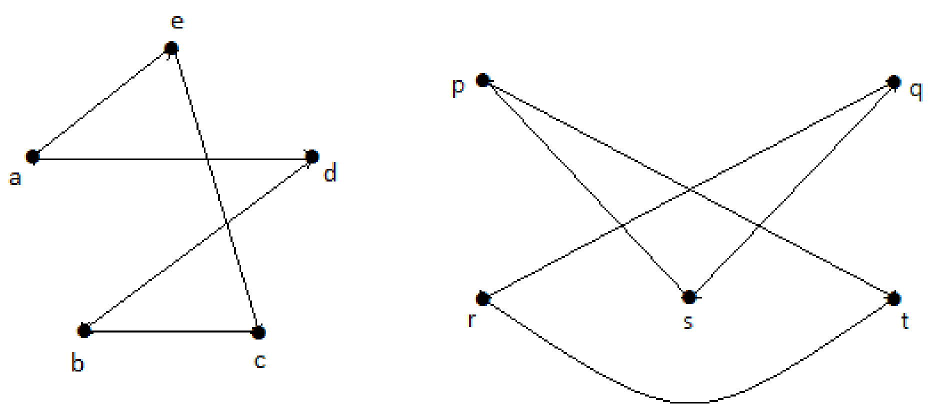


Fig 3.5

Here, (G1’ ≡ G2’), hence (G1 ≡ G2).

When graphs are not isomorphic?

- If G and G’ have different number of vertices, than the graphs are not isomorphic.

- If G and G' have different degree sequence, than the graphs are not isomorphic.
- Two graphs G and G' are isomorphic if the vertices of a given class in graph G must correspond to the vertices of the same class in the graph G'.

Example:
Show that which of the following graph are isomorphic: (pg 102)

Solution:

Deg (G)	Deg (G')
Deg(a) = 3	Deg(a') = 3
Deg(b) = 2	Deg(b') = 2
Deg(c) = 3	Deg(c') = 3
Deg(d) = 3	Deg(d') = 3
Deg(e) = 1	Deg(e') = 1

Each graph has 5 vertices and 6 edges. There is proper correspondence between the two graphs i.e. a – a', b – b', c – c', d – d', and e – e'. Therefore, the graphs are isomorphic.

Example:
Show that the following graphs G and G' are isomorphic. (pg 103)

Solution

Deg (G)	Deg (G')
Deg(a) = 3	Deg(a') = 3
Deg(b) = 2	Deg(b') = 2
Deg(c) = 3	Deg(c') = 3
Deg(d) = 3	Deg(d') = 3
Deg(e) = 1	Deg(e') = 1

Each graph has 5 vertices and 7 edges. There is proper correspondence between the two graphs i.e. a – a', b – b', c – c', d – d', and e – e'. Therefore, the graphs are isomorphic.

Example:
Show that the following graphs G and G' are isomorphic. (pg 103)

Solution:
Vertices of degree 2 are e & f in G and b' & e' in G'
So e – b'
f – e'

vertices of degree 3 are a & c in G and a' & d' in G'
a – a'
c – d'

vertices of degree 4 are b & d in G and f' and c' in G'
b – f'

d – f'

now, the vertices of degree 3 in G are a and c and they are adjacent in G and they are adjacent in G' while a' and d' are not adjacent tin G'

So, the given graphs are not isomorphic.

Example:
Show that the given graphs shown in the figure are not isomorphic. (pg 104)

Solution:
Both graphs have 8 vertices and 11 edges

Deg (G)	Deg (G')
Deg(a) = 3	Deg(a') = 3
Deg(b) = 2	Deg(b') = 3
Deg(c) = 3	Deg(c') = 2
Deg(d) = 3	Deg(d') = 3
Deg(e) = 3	Deg(e') = 3
Deg(f) = 2	Deg(f') = 3
Deg(g) = 3	Deg(g') = 2
Deg(h) = 2	Deg(h') = 2

Both the graphs have 5 vertices of degree 3 and 3 vertices of degree 2
In the graph G, edge g & c are adjacent while in G' they are not adjacent.
In the graph G' edge b' & f' are adjacent while in G they are not adjacent.

So, graphs G and G' are not isomorphic.

3.2 Homomorphism

Two graphs G1 and G2 are said to be homomorphic, if each of these graphs can be obtained from the same graph 'G' by dividing some edges of G with more vertices. Take a look at the following example:

Graph G Fig 3.6 (a)

Divide the edge 'rs' into two edges by adding one vertex.

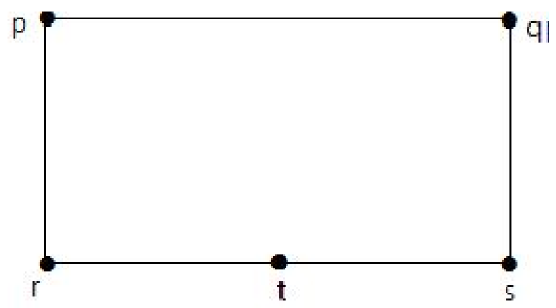


Fig 3.6 (b)

The graphs shown below are homomorphic to the first graph.

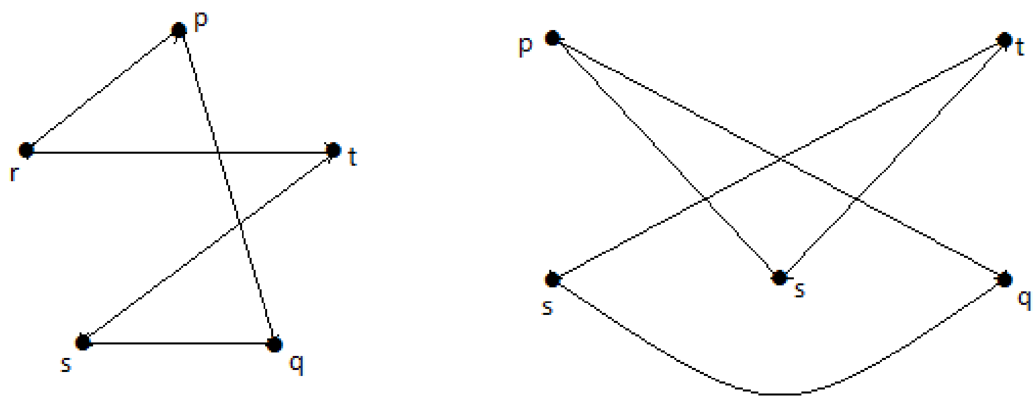


Fig 3.7

If G1 is isomorphic to G2, then G is homeomorphic to G2 but the converse need not be true.

3.3 Subgraphs

A graph g is said to be a subgraph of a graph G if all vertices and all the edges of g are in G, and each edge of g has the same end vertices in g as in G. The graphs in fig (b) is a subgraph of fig (a). The concept of subgraph is similar to the concept of subset in set theory. A graph g is subset of graph G is written as $g \subset G$.

1. Every graph is its own subgraph.
2. A subgraph of a subgraph of G is a subgraph of G
3. A single vertex in a graph G is a subgraph of G
4. A single edge in G, together with its end vertices, is also a subgraph of G.

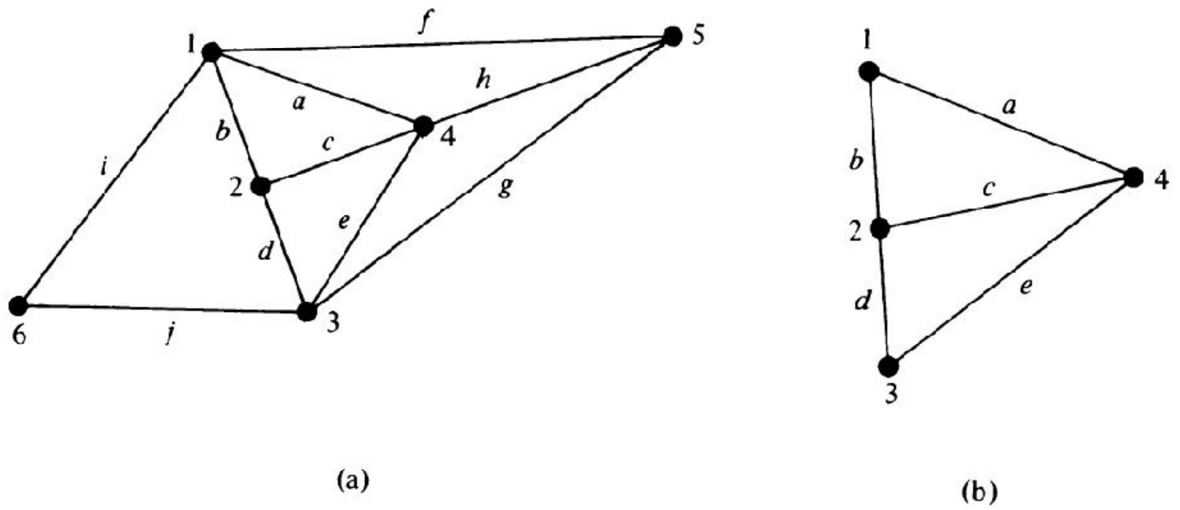


Fig 3.8

3.3.1 Edge-Disjoint subgraphs

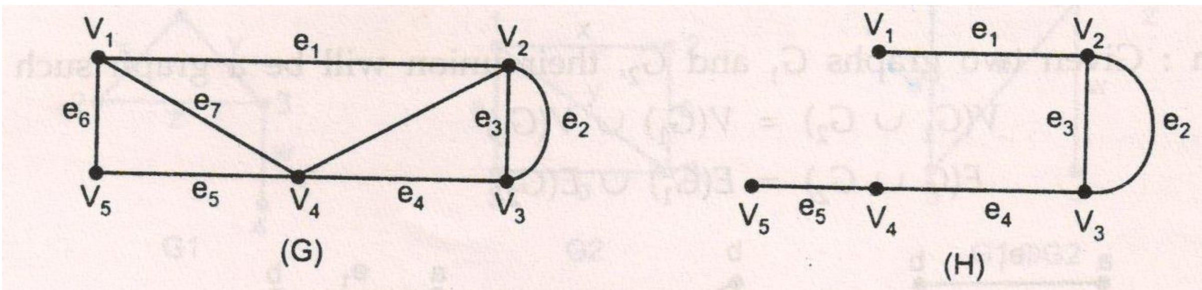
Two (or more) subgraphs g1 and g2 of a graph G are said to be edge-disjoint if g1 and g2 do not have any edge in common. Although edge-disjoint graphs do not have any edge in common, they may have vertices in common.

3.3.2 Vertex-Disjoint subgraph

Subgraphs that do not even have vertices in common are said to be vertex-disjoint. Graphs that have no vertices in common cannot possibly have edges in common.

3.3.3 Spanning subgraph

A subgraph of G is said to be spanning subgraph if it contains all the vertices of G. If $V(H) \subset V(G)$ and $E(H) \subset E(G)$ then H us proper subgraph of G and if $V(H) = V(G)$ then H is a spanning subgraph of G.



3.3.4 Vertex deleted and edge deleted subgraphs

The removal of vertex v_i from graph G results in to subgraph G-v_i of G containing of all vertices in G except vi and all edges not incident with *vi*.

The removal of an edge x_j from G yields the spanning subgraph G-x_j containing all edges except x_j.