Unit V
Connected and Disconnected graphs

5.1 Connected and Disconnected graphs

A graph is said to be **connected** if there exist at least one path between every pair of vertices otherwise graph is said to be **disconnected**. A null graph of more than one vertex is disconnected (Fig 3.12). Fig 3.9(a) is a connected graph where as Fig 3.13 are disconnected graphs.

A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a ***component***. The graphs in fig 3.13 consists of two components.

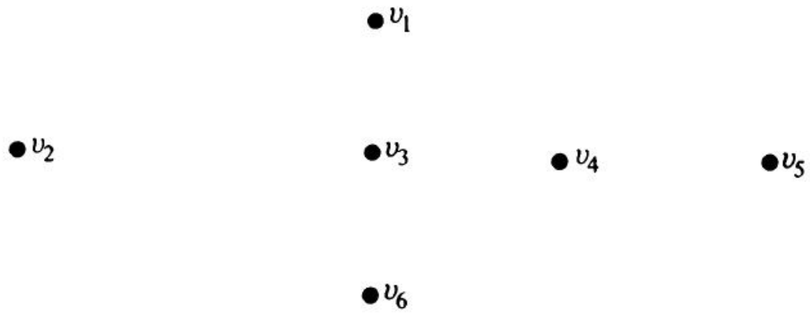

Fig 3.12: Null Graph of six vertices

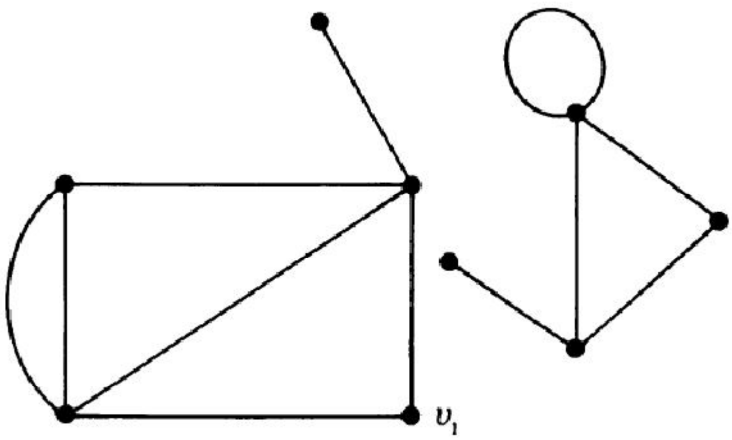

Fig 3.13: A disconnected graph with two components

5.2 Theorem1:

A graph G is said t be disconnected if and only if its vertex V can be portioned into two nonempty, disjoint subsets v1 and v2 such that there exists no edge in G whose one end vertex is in subset v1 and the other in subset v2.

Proof:

Suppose that such a partitioning exists. Consider two arbitrary vertices ***a*** and ***b*** in **G**, such that ***a*** ∈ *V1* and ***b*** ∈ *V2*. No path can be exists between vertices ***a*** and ***b***; otherwise there would be at least one edge whose one end vertex would be in *V1* and the other in *V2*. Hence if partition exists, **G** is not connected.

Conversely, let **G** be disconnected so there exists a vertex ***a*** in **G** and a vertex ***b*** in **G** such that there is no path between ***a*** and ***b*** in **G**

Let ***V1*** be the set of all vertices that are joined by path to ***a***. Since **G** is disconnected ***V1*** does not include all vertices of **G**

Let $V2 = V - V1$ then $V1 \cap V2 = \phi$ and $V1 \cup V2 = V$

Thus the remaining vertices will form a (non-empty) set of ***V2*** i.e. ***b*** ∈ ***V2***. Thus no vertex in ***V1*** is joined to any vertex in ***V2*** by an edge.

5.3 Theorem2:

If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof:

Let **G** be a graph with all even degree of vertices except two vertices ***v1*** and ***v2***, which are odd degree. We know that number of vertices with odd degree in a graph is always an even. That is no graph can have an odd number of odd vertices. Therefore, in graph **G**, ***v1*** and ***v2*** must be belong to the same component and hence must have a path between them.

5.4 Theorem3:

A simple graph (i.e., a graph without parallel edges or self-loops) with n vertices and k components can have at most $(n - k)(n - k + 1)/2$ edges.

Proof:

Proof. Let the number of vertices in each of the k-components of a graph be n_1, n_2, n_3 ... n_k. Thus

$$n_1 + n_2 + \ldots n_k = n \quad \text{or} \quad \sum_{i=1}^{k} n_i = n \quad 1 \le i \le k$$

Suppose a component with n_i vertices, then maximum number of possible edges

$$= \frac{n_i(n_i - 1)}{2}, \text{ when it is complete.}$$

Hence, the maximum number of edges are

$$\frac{1}{2}\sum_{i=1}^{k} n_i(n_i - 1) = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i \quad \ldots(i)$$

Now, we have

$$\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k$$

i.e.,

$$\sum_{i=1}^{k}(n_i - 1) = n - k, \text{ squaring both sides.}$$

$$\left[\sum_{i=1}^{k}(n_i - 1)\right]^2 = n^2 + k^2 - 2nk.$$

or

$$\sum_{i=1}^{k}(n_i^2 - 2n_i) + k + \text{non-negative cross terms} = n^2 + k^2 - 2nk$$

or

$$\sum_{i=1}^{k} n_i^2 \le 2n - k + n^2 + k^2 - 2nk$$

or

$$\sum_{i=1}^{k} n_i^2 \le n^2 - (k - 1)(2n - k)$$

Put this value in equation (i), we get

$$\frac{1}{2}\sum_{i=1}^{k} n_i(n_i - 1) \le \frac{1}{2}[n^2 - (k-1)(2n-k)] - \frac{1}{2}n$$

$$= \frac{1}{2}[n^2 - 2nk - k^2 + n - k]$$

$$= \frac{1}{2}[n-k][n-k+1]$$

5.5 Example:

Prove that a simple graph with n vertices must be connected if it has more than [(n - 1) (n – 2)]/2 edges.

Solution:

Suppose G is a simple graph with *n* vertices

Choose *(n – 1)* vertices such that $v_1, v_2, v_3, ...v_{n-1}$ of G.

We know that the maximum number of edges in a simple graph with *n* vertices is *n(n-1)/2*.

So we *[(n – 1) (n – 2)]/2* number of edges can be drawn for *(n – 1)* vertices. Thus if we have more than *[(n – 1) (n – 2)]/2* edges than at least one edge should be drawn between the n^{th} vertex i.e. v_n to some vertex v_i.

Hench G must be connected.

5.6 Example

Let G be a disconnected graph with n vertices where n is even. If G has two components each of which is complete, prove the G has a minimum of n(n – 1)/4 edges.

Solution

Let x be the number of vertices in one of the components than the other component has (n – x) vertices. Since both components are complete graph.

We know that number of edges in a simple graph with n vertices are n(n – 1)/2.

Thus, the number of edges with x and (n – x) vertices are

$$[x(x-1)]/2 \quad \text{and} \quad [(n-x)(n-x-1)]/2$$

So, the total numbers of edges are

$$E = \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2}$$

$$= \frac{x^2 - x + n^2 - nx - n - nx + x^2 + x}{2}$$

$$E = x^2 - nx + \frac{n}{2}(n-1) \text{ -- (i)}$$

Differentiate w.r.t. x, we get

$$E' = 2x - n$$

Again differentiate w.r.t. x, we get

E’’ = 2 > 0

Therefore, E is minimum when 2x – n > 0

i.e. $x = \frac{n}{2}$

For minimum value of n, put the value of x in equation (i)

$$E = \left(\frac{n}{2}\right)^2 - n\left(\frac{n}{2}\right) + \frac{n}{2}(n-1)$$

$$= \frac{[n(n-1)]}{4}$$